Connectivity of nodes

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BT exact

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TYPESET IN PDFLATEX ON A LINUX SYSTEM

Contents

Introduction

I consider connectivity of nodes with a radio range ρ placed uniformly and randomly in a bounded region under various models:

- *Poisson 1d model*: the nodes exist on all of $\mathbb{R}$ with a exponential distribution of separation with parameter λ, and a window of unit length is placed over them. The number of nodes visible through the window is Poisson distributed.
- *fixed-n 1d model*: there are exactly n nodes independently and uniformly placed in $[0, 1]$.
- *Poisson 2d model*: the nodes exist on all of $\mathbb{R}^2$ with a intensity λ, and a finite-area window is placed over them. The number of nodes visible through the window is Poisson distributed.
- *fixed-n 2d model*: there are exactly n nodes independently and uniformly placed in a bounded region R.

Notation:

- ▷ *pdf=probability density function*
- ▷ *cdf=cumulative distribution function*
- ▷ *The notation is sloppy in not distinguishing a RV X and its values x*
- ▷ $[\![x]\!]$ *is the indicator function: 1 if x is true, else 0*

Theory for the Poisson 1d model

- λ is the intensity of nodes per unit length
- The pdf of the internode distance d is $f(d) = \lambda e^{-\lambda d}$
- The cdf of the internode distance is $F(d) = 1 - e^{-\lambda d}$
- The expectation of d is $\mathbb{E}[d] = 1/\lambda$

More theory for the Poisson 1d model

We now place a unit length window over $\mathbb{R}$ and assume that n nodes are visible. The following results are conditional on n

- There are $n-1$ internode intervals, and the cdf of the maximum interval is $F_{n-1}(d) = (1-e^{-\lambda d})^{n-1}$
- the cdf of the minimum interval is $F_1(d) = 1-e^{-n\lambda d}$
- the pdf of the minimum interval is $f_1(d) = n\lambda e^{-n\lambda d}$
- The expectation of the minimum interval is $\mathbb{E}[d_{(1)}] = 1/(2\lambda)$, so is half the expectation of the internode distance
- The intervals have correlation $-1/n$
- The probability of full connectivity for the n nodes is thus approximately (i.e. ignoring correlation and edge effects) $F_{n-1}(\rho) = (1-e^{-\lambda\rho})^{n-1}$
- This result is only approximate. We should expect deviations small n

The exact theory for the fixed-n case is here

Order statistics theory [dav70]

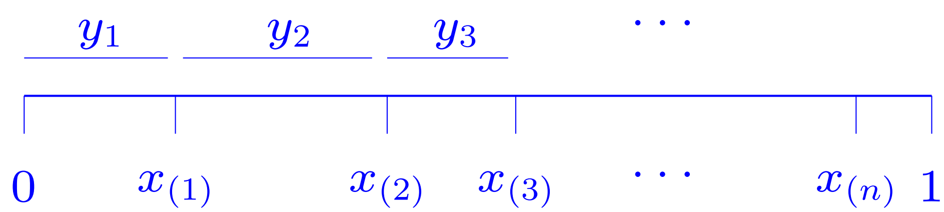
Let $x_1, x_2, \ldots, x_n$ be RVs uniformly distributed in $[0,1]$.

Sort them in increasing order as $x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}$.

- The pdf of $x_{(k)}$ is
$$\frac{1}{(k-1)!}\binom{n}{k}x^{k-1}(1-x)^{n-k}$$
- The cdf of the range $r = x_{(n)} - x_{(1)}$ is $nr^{n-1} - (n-1)r^n$
- If $w_{rs} = x_{(s)} - x_{(r)}$, then the pdf of w_{rs} is
$$w_{rs}^{s-r-1}(1-w_{rs})^{n-s+r}/\mathrm{B}(s-r, n-s+r+1)$$
- For the special case of adjacent nodes ($s = r+1$), this becomes $n(1-w_{r,r+1})^{n-1}$, which gives a cdf of $1-(1-w_{r,r+1})^n$
- However, the $w_{r,r+1}$ are not independent random variables, so the probability that the maximum of $n-1$ samples of $w_{r,r+1}$ is less than a constant ρ, is NOT $[1-(1-\rho)^n]^{n-1}$
  - ▷ *But this is approximately correct for large n and ρ near 1 and is plotted in blue on the graphs of simulation results*
  - ▷ *As $\rho \to 1$, this becomes $1-(n-1)(1-\rho)^n$. cf. the exact equation*

Exact theory for the fixed-n 1d model

- Let $y_k = x_{(k)} - x_{(k-1)}$ be the gaps ($k = 2, \ldots, n$), with $y_1 = x_{(1)}$



- Their joint pdf is (for $1 \leqslant m \leqslant n$ and $\sum_{i=1}^{m} y_i \leqslant 1$)

$$f(y_1, y_2, \ldots, y_m) = \frac{n!}{(n-m)!}\left(1 - \sum_{i=1}^{m} y_i\right)^{n-m}$$

- If c_i are constants such that $\sum_{i=1}^{m} c_i \leqslant 1$, then by integrating the pdf we obtain

$$\Pr\left[y_1 > c_1, y_2 > c_2, \ldots\right] = \left(1 - \sum_{i=1}^{m} c_i\right)^{n-1}$$

- Boole's law for the probability of at least one event A_i of n events $A_1, A_2, \ldots, A_n$ occurring is

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i} \Pr\left[A_i\right] - \sum_{i<j}\sum \Pr\left[A_i A_j\right] + \cdots + (-1)^{n-1} \Pr\left[A_1 A_2 \ldots A_n\right]$$

Exact theory for the fixed-n 1d model (cotd.)

- We don't care about y_1, so we put $c_1 = 0$
- Using Boole's law, the probability that the largest y_k exceeds some constant ρ is

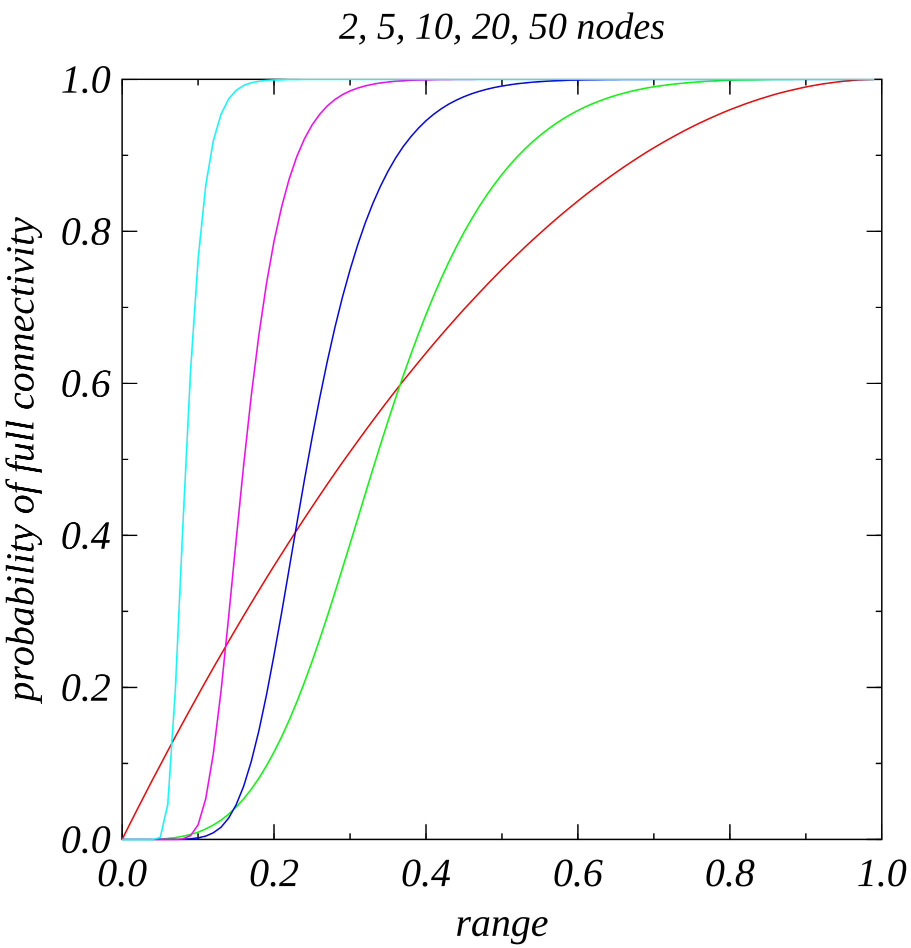
$$\mathrm{Pr}\left[y_{(n)} > \rho\right] = (n-1)\,\mathrm{Pr}\left[y_1 > \rho\right] - \binom{n-1}{2}\mathrm{Pr}\left[y_1 > c_1, y_2 > c_2\right] + \ldots$$

- Thus

$$\boxed{\mathrm{Pr}\left[\text{fully connected}\right] = 1 - \sum_{i=1}^{\lfloor 1/\rho \rfloor} (-1)^{i+1}\binom{n-1}{i}(1-i\rho)^n}$$

- This is plotted as a red line on the following pages
- Note that for $\rho > 1/2$, this is exactly $1-(n-1)(1-\rho)^n$.
  - ▷ *cf. an approximation*

Probability of connectivity for the fixed-n 1d model



2 5 10 20 50

Theory for the Poisson 2d model [cre91]

- λ is the intensity of nodes per unit area
- The pdf of the nearest neighbour distance d is $f(d)=2\pi\lambda d e^{-\lambda\pi d^2}$
- The cdf of d is $F(d)=1-e^{-\pi\lambda d^2}$
- The expectation of d is $\mathbb{E}[d]=1/(2\lambda^{1/2})$
- The variance of d is $(4-\pi)/(4\pi\lambda)$
- The probability of a node being isolated (i.e. having no neighbour within range ρ) is $e^{-\pi\lambda\rho^2}$

Theory for the Poisson 2d model

We now place a window R of area A over $\mathbb{R}^2$

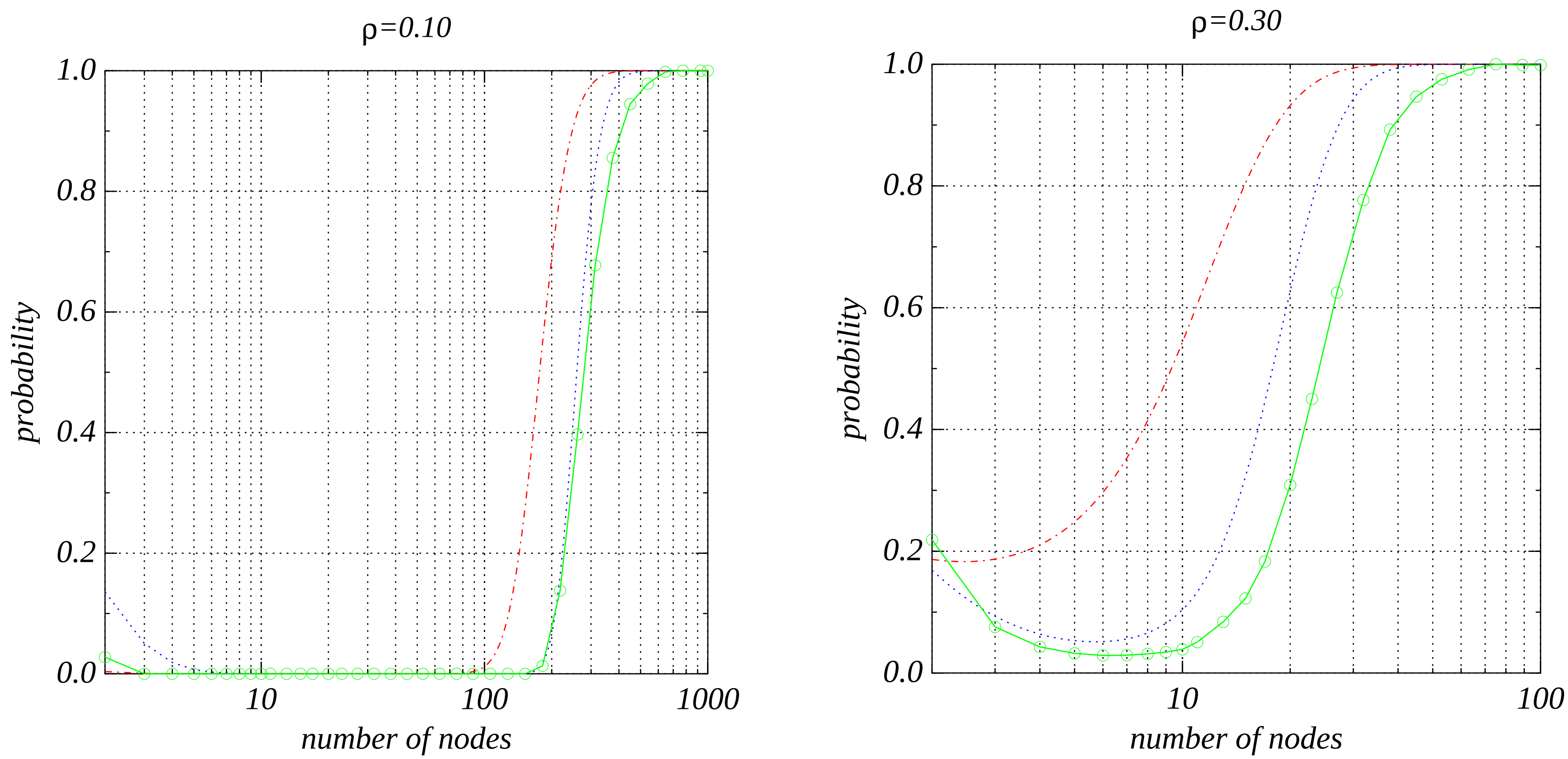
- The number of nodes visible will be Poisson distributed with mean λA

- Conditional on n nodes being visible, and if the nearest neighbour distances were independent (which is *not* the case) the probability of no node being isolated would be $\left(1-e^{-\pi\lambda\rho^2}\right)^n$

- There is no simple way to compute the probability of full connectivity. However, since a necessary condition is that no node is isolated, the last expression is an approximate upper bound for the fixed-n model and is plotted in red on the following graphs

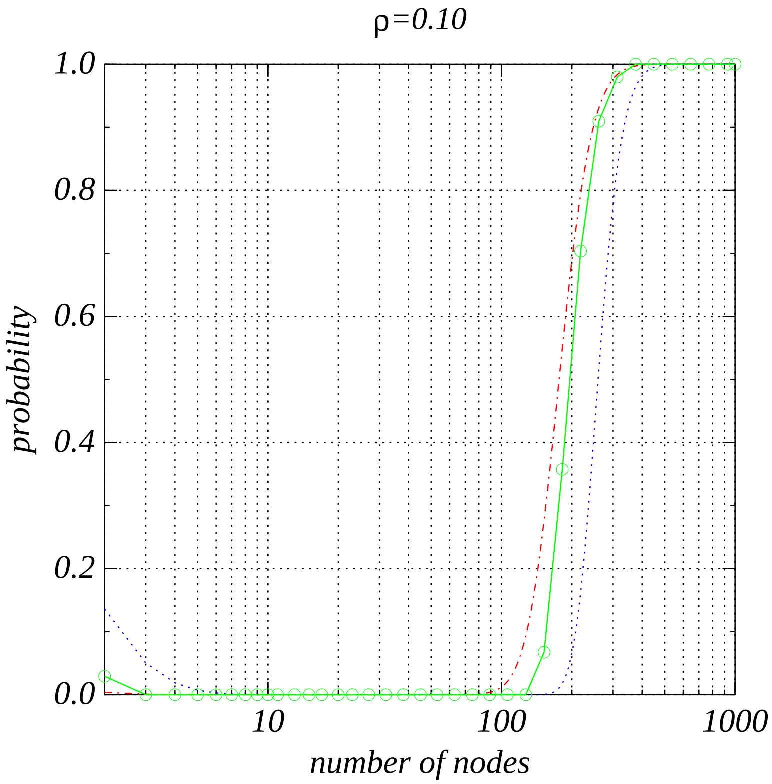
- The blue curve is the asymptotic probability of the whole region R being covered, using this theory

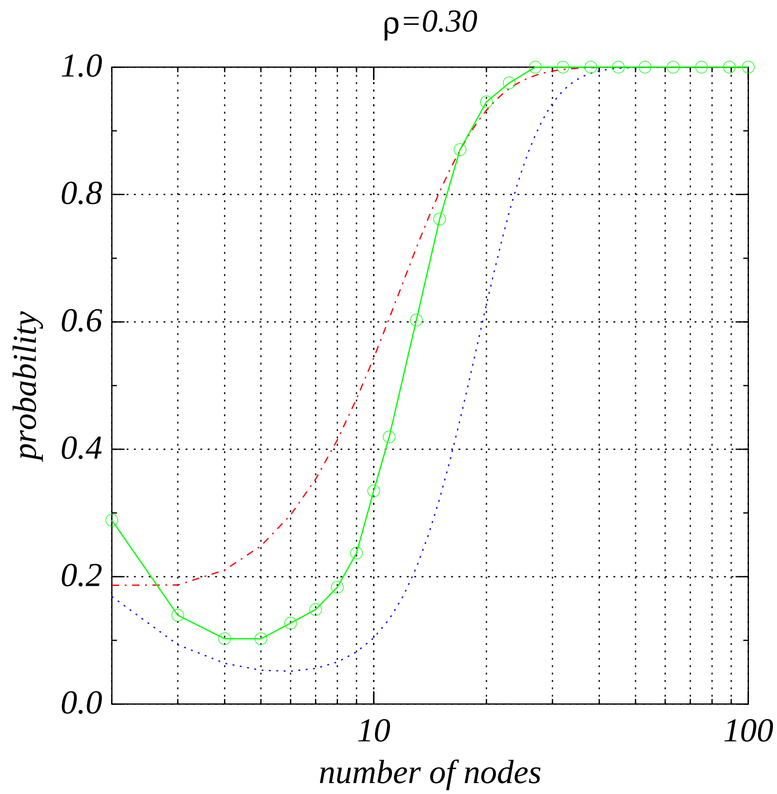
Simulation results - square, $\rho = 0.1, 0.3$



$\left[1-e^{-\pi\lambda\rho^2}\right]^n$ simulation asymptotic

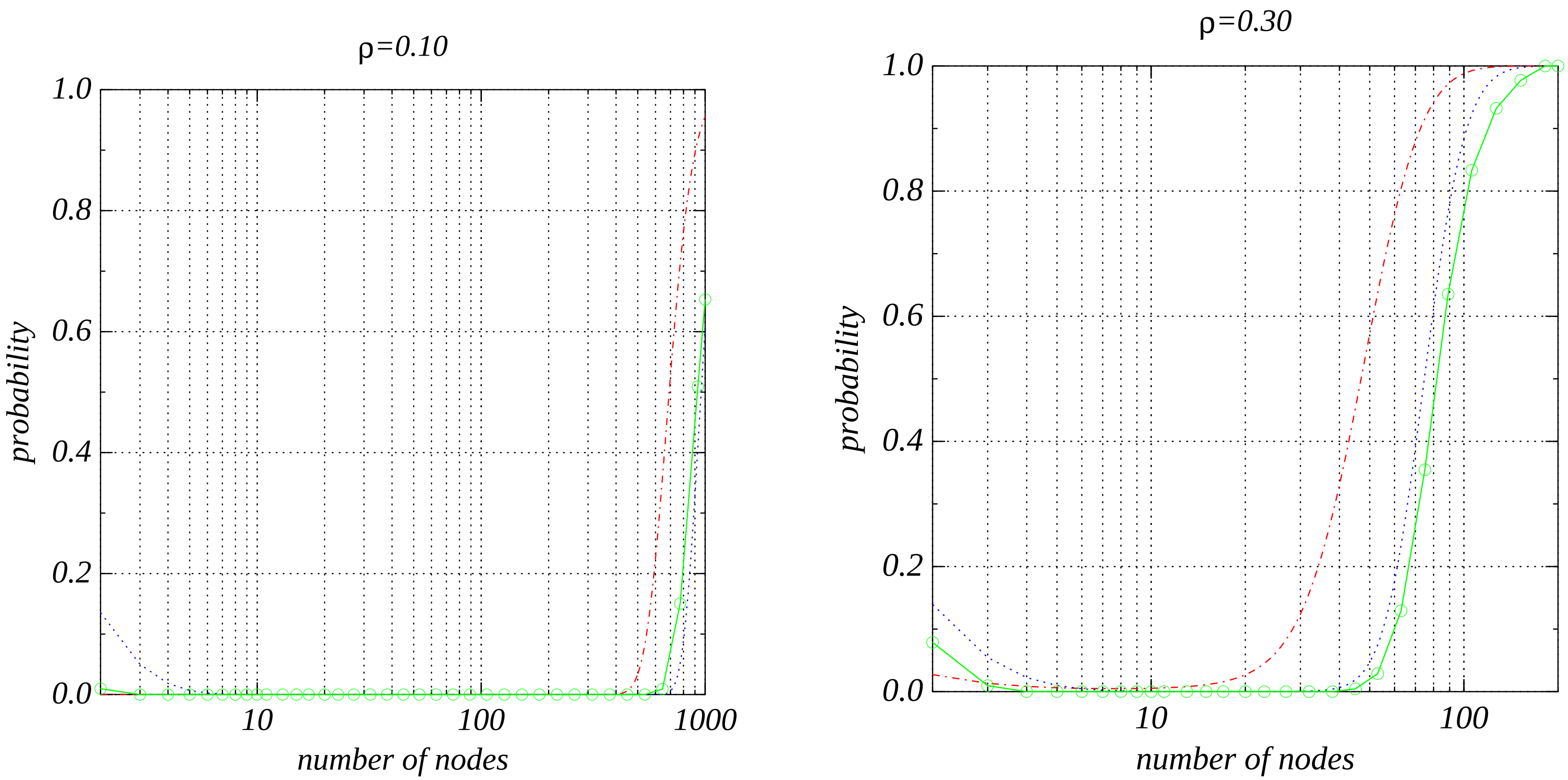
Simulation results - torus, $\rho = 0.1, 0.3$





$\left[1 - e^{-\pi\lambda\rho^2}\right]^n$ simulation asymptotic

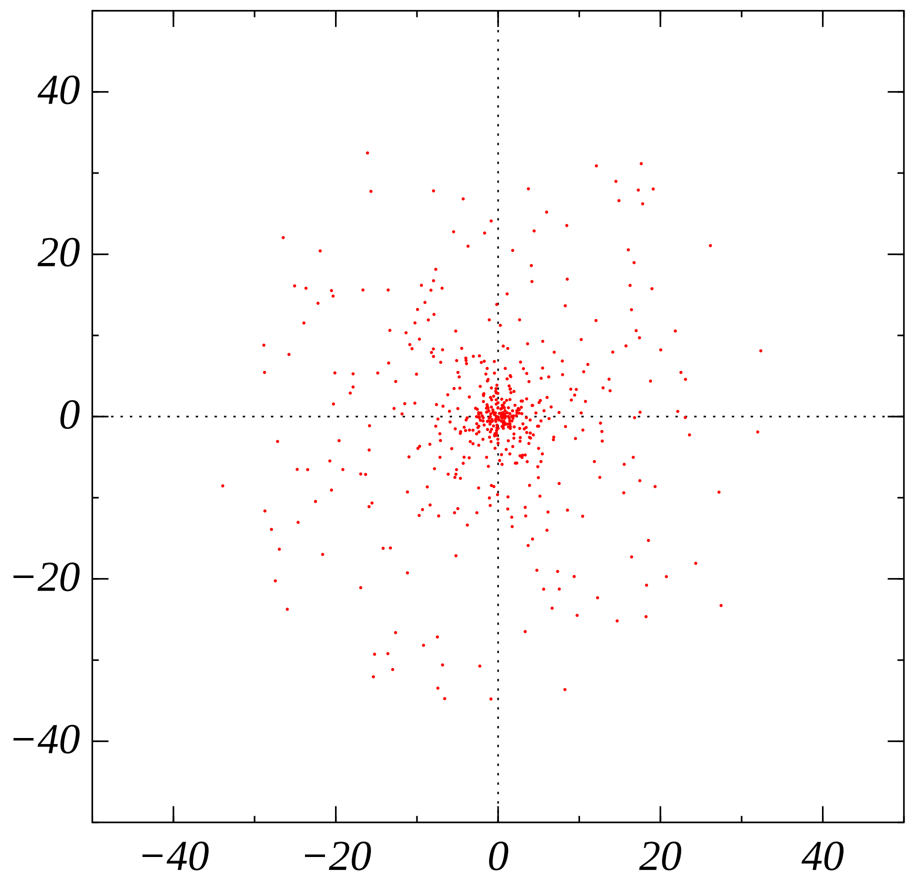
Simulation results - unit-radius disk, $\rho = 0.1, 0.3$



$\left[1 - e^{-\pi\lambda\rho^2}\right]^n$ simulation asymptotic

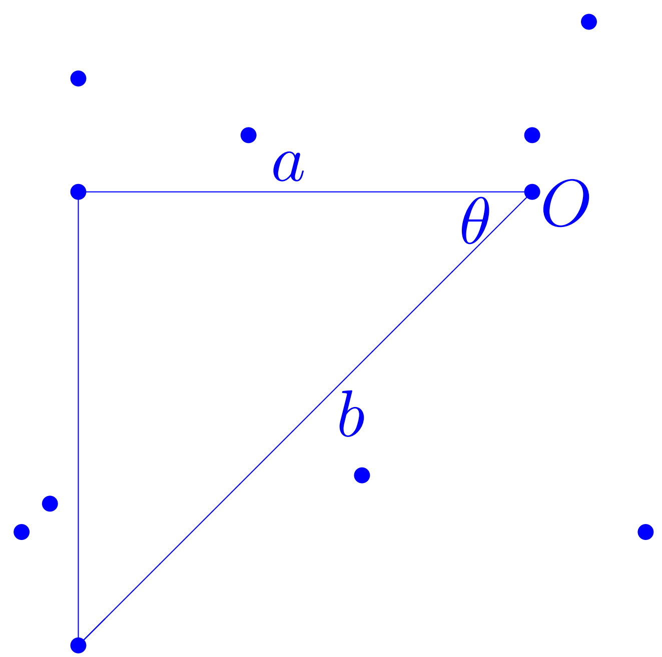
Non-homogeneous Poisson process

Example: intensity falls off exponentially from an access point



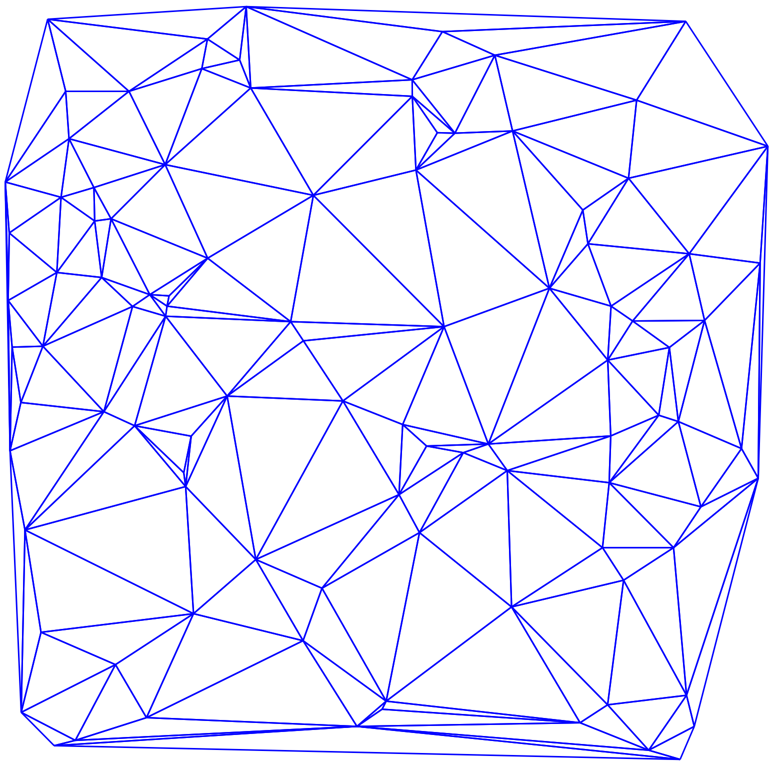
Delaunay triangles 1 [kla90]

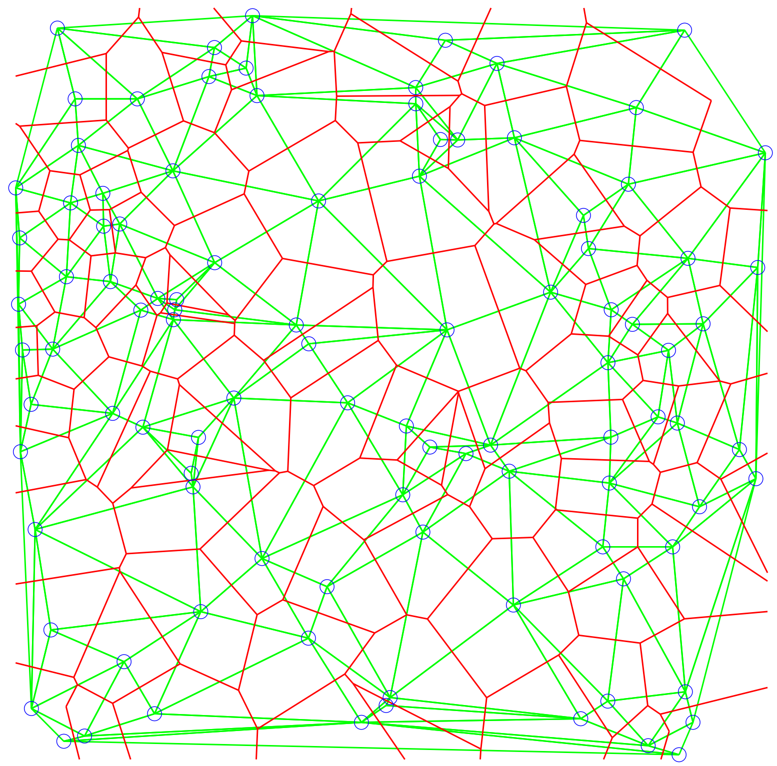
- Pick one node O from a planar Poisson process of intensity λ
- Consider triangles formed by two other nodes
- Call it *empty* if no other nodes are in the triangle
- Call it *very empty* if no other nodes are in the circumcircle of the triangle
- An empty triangle:



Delaunay triangles 2

The Delaunay triangulation consists of very empty triangles only. The second figure shows the Voronoi tesselation superimposed.





Delaunay triangles 3

- Let a and b be the lengths of the two edges adjacent to O and θ the angle
- For very empty triangles, the joint pdf is

$$2\pi ab\lambda^4 \exp\left[-\pi\lambda^2 \frac{a^2+b^2-2ab\cos\theta}{4\sin^2\theta}\right]$$

- For very empty triangles, the pdf of the area A is $\lambda^2 A\exp(-\lambda A)$
- For very empty triangles, the mean of a is $\frac{32}{9\pi\lambda}$
- For empty triangles, the pdf is $2\pi ab\lambda^4 \exp\left[-\lambda^2 ab\sin(\theta)/2\right]$
- In both cases, the mean number of triangles at O is 6

Delaunay triangles 4

Integrating out over side b and angle θ, we get for the pdf of side a:

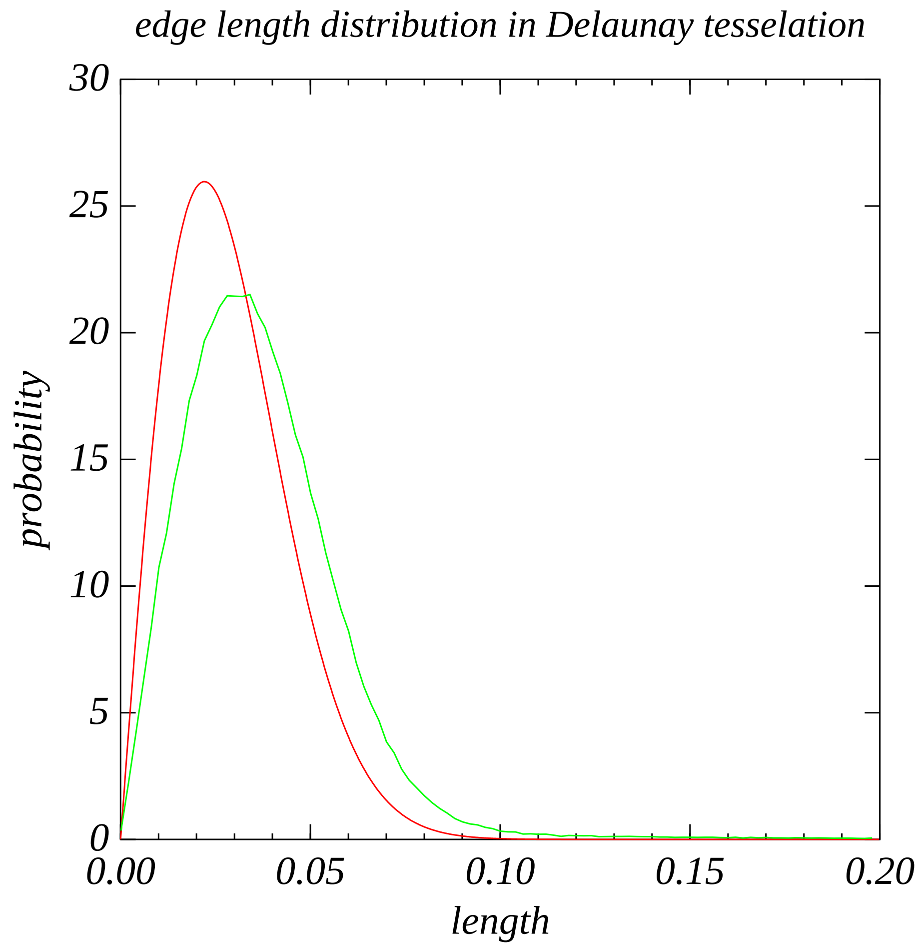
$$4a\lambda \int_0^\pi \sin^2\theta \exp\left[\frac{-\pi a^2\lambda}{4\sin^2\theta}\right] \left[1+e^{\alpha^2\nu^2}\alpha|\nu|\pi^{1/2}(\mathrm{erf}(\alpha|\nu|)+\mathrm{sign}(\nu))\right] d\theta$$

where

$$\alpha = \frac{\sin\theta}{(\pi\lambda)^{1/2}}$$

$$\nu = \frac{a\lambda\cos\theta}{2\sin^2\theta}$$

Delaunay triangles 5



1000 nodes: exact simulation

Distance distribution for some regions

Two points independently uniformly distributed in a region R; the pdf of the distance d is $f(d)$, mean distance is μ:

- $R =$ unit interval, $f(d) = 2(1-d)[[0 \leqslant d \leqslant 1]]$, $\mu = 1/3$
- $R =$ 1-torus, $f(d) = 2[[0 \leqslant d \leqslant 1/2]]$, $\mu = 1/4$
- $R =$ 2-torus

$$f(d) = \begin{cases} 2\pi d & \text{if} \quad 0 \leqslant d < 1/2 \\ 2d\left[\pi - 4\sec^{-1}(2d)\right] & \text{if} \quad 1/2 \leqslant d \leqslant \sqrt{2} \end{cases}$$

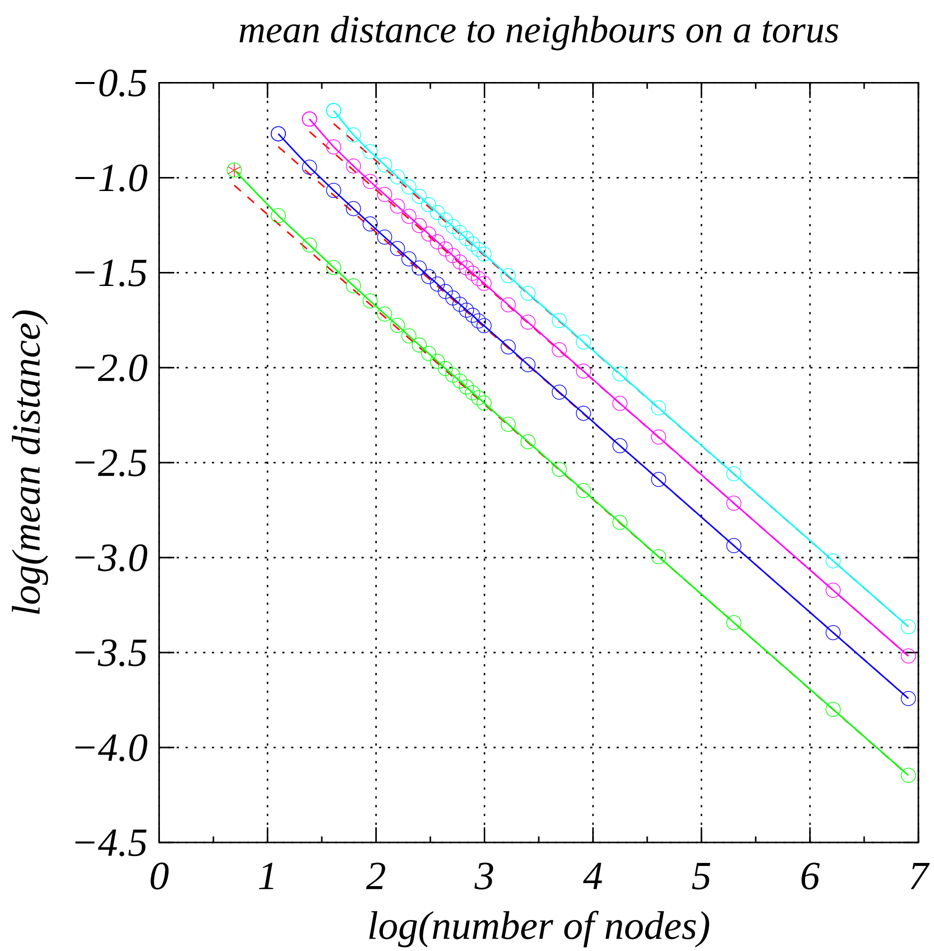
$$\mu = \left[\sqrt{2} + \log(1+\sqrt{2})\right]/6$$

- $R =$ unit radius disk, $f(d) = d/\pi\left[4\arctan\left(\sqrt{4-d^2}/d\right) - d\sqrt{4-d^2}\right][[0 \leqslant d \leqslant 2]]$, $\mu = 128/(45\pi)$
- $R =$ unit sphere, $\mu = 36/35$
- $R =$ unit square, see next slide

Asymptotics for near neighbours

- Put n points in a unit torus in $\mathbb{R}^2$
- Let $d_k =$ be the distance to kth nearest neighbour
- Then it is known that ([eva02]): $\mathbb{E}[d_k] = \pi^{-1/2} \, \frac{\Gamma(k+1/2)}{\Gamma(k)} \, n^{-1/2} + \mathcal{O}(n^{-3/2})$
- So $\mathbb{E}[d_1] = 1/2 \, n^{-1/2} + \mathcal{O}(n^{-3/2})$

Asymptotics for nearest neighbours - simulations



nearest, second nearest, . . .

asymptotic, $*$ is exact value for $n = 2, k = 1$, namely $[2^{1/2}+\log(1+2^{1/2})]/6$

Exact theory for mean distances on a torus 1

Recall that our pdf and cdf are defined piecewise: I will use $^<$ and $^>$ to indicate the pieces on $[0,1/2]$ and $[1/2,1/\sqrt{2}]$ respectively:

$$
\begin{aligned}
f^<(x) &= 2\pi x \\
f^>(x) &= 2x\left[\pi - 4\sec^{-1}(2x)\right] \\
F^<(x) &= \pi x^2 \\
F^>(x) &= \sqrt{4x^2-1} + x^2\left[\pi - 4\sec^{-1}(2x)\right]
\end{aligned}
$$

Exact theory for mean distances on a torus 2

I will use the subscript $k\!:\!n$ to denote the kth order statistic in a sample of size n

Thus

$$f^{<}_{k:n}(x) \;=\; \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}\, f^{<}(x)\,[F^{<}(x)]^{k-1}\,[1-F^{<}(x)]^{n-k}$$

and similarly for $f^{>}_{k:n}(x)$.

So we have

$$f_{k:n}(x) = f^{<}_{k:n}(x)[\![0 \leqslant x \leqslant 1/2]\!] + f^{>}_{k:n}(x)[\![1/2 < x \leqslant 1/\sqrt{2}]\!]$$

and

$$\mu_{k:n} = \mu^{<}_{k:n} + \mu^{>}_{k:n}, \quad 1 \leqslant k \leqslant n$$

Exact theory for mean distances on a torus 3

To get the mean, we can do the lower integral exactly:

$$\begin{aligned}\mu^{<}_{k:n} &= \int_0^{1/2} t\, f^{<}_{k:n}(t)\, dt \\ &= \frac{(\pi/4)^k}{(2k+1)} \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \mathsf{F}\left(\begin{matrix} k+1/2 \;\; k-n \\ k+3/2 \end{matrix} \middle| \, \pi/4\right)\end{aligned}$$

but the upper integral

$$\mu^{>}_{k:n} = \int_{1/2}^{1/\sqrt{2}} t\, f^{>}_{k:n}(t)\, dt$$

will have to be approximated. Luckily, it is typically a very small correction term to $\mu^{<}_{k:n}$, and goes to zero geometrically with n.

Exact theory for mean distances on a torus 4

Because $k-n<0$, the hypergeometric function above is a terminating series:

$$\mathsf{F}\left(\begin{matrix}k+1/2 & k-n\\ k+3/2 & \end{matrix}\middle|\,\pi/4\right)=\sum_{i=0}^{n-k}\frac{(k+1/2)^{\bar{i}}(k-n)^{\bar{i}}}{(k+3/2)^{\bar{i}}}\frac{(\pi/4)^i}{i!}$$

It is quite feasible to evaluate $\mu_{k:n}$ exactly from this, but if desired we can use an integral representation of this function and Watson's lemma to find the large n asymptotics. I omit all the details of this. The results are on the next page.

Exact theory for mean distances on a torus 5

We now do asymptotics ($n \to \infty$) for the mean distance to the kth neighbour

$$\begin{aligned}
\mu^{<}_{1:n} &\sim n \left[\frac{1}{2}n^{-3/2} - \frac{3}{16}n^{-5/2} + \frac{25}{256}n^{-7/2} - \frac{105}{2048}n^{-9/2} + \dots\right] \\
\mu^{<}_{2:n} &\sim n^{2} \left[\frac{3}{4}(n-1)^{-5/2} - \frac{45}{32}(n-1)^{-7/2} + \frac{1155}{512}(n-1)^{-9/2} - \dots\right] \\
\mu^{<}_{3:n} &\sim n^{3} \left[\frac{15}{16}(n-2)^{-7/2} - \frac{525}{128}(n-2)^{-9/2} + \dots\right] \\
\mu^{<}_{4:n} &\sim n^{4} \left[\frac{35}{32}(n-3)^{-9/2} - \dots\right] \\
&\dots \\
\mu^{<}_{k:n} &\sim \Gamma(k+1/2)/\Gamma(k)\, n^{-1/2}
\end{aligned}$$

Note: for the 2d Poisson process, we have $\frac{1}{2}n^{-1/2}$ exactly for the nearest neighbour ($k=1$)

Exact theory for mean distances on a torus 6

To compute the contribution to the mean from the upper integral, we need to do:

$$\mu^{>}_{k:n} = \int_{1/2}^{1/\sqrt{2}} t\, f^{>}_{k:n}(t)\, dt$$

I do not know a way of approximating this for all n and k, but by making a series expansion of $f^{>}_{1:n}$ around $1/\sqrt{2}$ and just keeping the first term, for the nearest neighbour we get:

$$\mu^{>}_{1:n} \approx (3-2\sqrt{2})^n$$

Thus a good approximation for the mean distance to the nearest neighbour is

$$\mu_{1:n} = \mu^{<}_{1:n} + \mu^{>}_{1:n} \sim 1/2\, n^{-1/2} - 3/16\, n^{-3/2} + (3-2\sqrt{2})^n$$

Almost sure connectivity results [mil70]

- Planar process of intensity λ in region R

- Let $p(r) = \text{Pr}\left[\text{every point of } R \text{ covered by a disk radius } r\right]$

- Then, as $|R| \to \infty$

$$p(r) \sim \exp\left[-\lambda|R|\,e^{-\pi\lambda r^2}(1+\pi\lambda r^2)\right]$$

- This is plotted in blue on these graphs of simulation results

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